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On rational Drinfeld associators

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Abstract We prove an estimate on denominators of rational Drinfeld associators. To obtain this result, we prove the corresponding estimate for the p-adic associators stable under the action of suitable elements of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. As an application, we settle in the positive Duflo's question on the Kashiwara–Vergne factorizations of the Jacobson element $J_p(x, y) = (x + y)^p - x^p - y^p$ in the free Lie algebra over a field of characteristic p. Another application is a new estimate on denominators of the Kontsevich knot invariant.

Keywords Associators · Grothendieck-Teichmüller group · Knot invariants

Mathematics Subject Classification (2010) 17B37 · 57M27

1 Introduction

Drinfeld associators were defined in [6] and play an important role in many fields of mathematics, including number theory (for recent developments, see [11]), low-dimensional topology [4,5,16], Lie theory [8], and deformation quantization [18]. In

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literature, there are two examples of Drinfeld associators defined by explicit formulas. The Knizhnik–Zamolodchikov associator Φ_{KZ} of [6] is defined over $\mathbb{C}$ and expressed in terms of iterated integrals and multiple zeta values. An explicit example over $\mathbb{R}$ is given in [3] and [20] in terms of Kontsevich integrals over configuration spaces.

The existence of rational associators was proved in [6]. Constructing an explicit example is considered to be a major problem in associator theory, and (to the best of our knowledge) it remains open.

In this paper, we obtain estimates on denominators of rational associators. More precisely, we define the set of *natural* rational associators with the property that the denominator in its degree n component is a divisor of

$$D(n) = \prod_{\substack{p \ \text{prime} \\ p \le n+1}} p^{b_p(n)}, \tag{1}$$

where

$$b_p(n) = \left[\frac{p}{(p-1)^2} n - \frac{1}{p-1} \right],$$

and $[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$. Our main result is Theorem 4.2, which shows that the set of natural associators is non-empty.

For a natural associator, the prime $p > 2$ appears for the first time in the denominator of the component of degree $p - 1$. We show that this part of the estimate is optimal. We use this observation to settle in the positive Duflo's question [7] on Kashiwara–Vergne factorizations of the Jacobson element.

In more detail, the Kashiwara–Vergne problem [12] in Lie theory is to find a factorization of the Campbell–Hausdorff series in the form

$$x + y - \ln(e^y e^x) = (1 - \exp(-\operatorname{ad}_x))A(x, y) + (\exp(\operatorname{ad}_y) - 1)B(x, y),$$

where $A(x, y)$ and $B(x, y)$ are Lie series, which satisfy an additional linear equation with coefficients given by Bernoulli numbers (for details see e.g. [19]). The factor p^{-1} appears for the first time in $\ln(e^y e^x)$ in degree p. The corresponding residue is the Jacobson element $J_p(x, y) = (x + y)^p - x^p - y^p$. Hence, it is natural to conjecture that $A(x, y)$ and $B(x, y)$ are p-integral up to degree $p - 2$ and have a simple pole in degree $p - 1$. If this is the case, one obtains a decomposition of the Jacobson element

$$J_p(x, y) = [x, a(x, y)] + [y, b(x, y)] \mod p,$$

where $a(x, y)$ and $b(x, y)$ are residues of $A(x, y)$ and $B(x, y)$, respectively. Finding such decompositions (with an extra constraint similar to the one in the Kashiwara–Vergne problem) is the Duflo's question.

Another property of natural associators is that they have non-zero convergence radius in the p-adic norm for every prime p.

Our strategy for proving Theorem 4.2 is as follows. There is a classical Galois theory result of Drinfeld stating that the Grothendieck–Teichmüller group character

$\chi : \mathrm{GT}_p \to \mathbb{Z}_p^*$ is surjective. If we choose an element $g_p \in \mathrm{GT}_p$ such that $\chi(g_p)$ generates a dense subgroup of $\mathbb{Z}_p^*$, then the unique p-adic associator fixed by g_p satisfies the required denominator estimates. We use these p-adic associators for all primes p to prove existence of a rational associator with the same estimates on denominators.

In [13] Kontsevich constructed a universal finite type invariant of knots in $\mathbb{R}^3$. For a knot K, this invariant is denoted $I(K)$, and it takes values in the graded algebra of chord diagrams $\mathrm{Chord}(\mathbb{Q})$. There is a combinatorial construction of $I(K)$, which uses an arbitrary Drinfeld associator [4,16]. It is remarkable that the final result is independent of the associator used in the computation. In particular, by choosing a natural rational associator we obtain a new estimate (Theorem 5.3) on denominators of $I(K)$,

$$I(K) \in \sum_{n=0}^{\infty} D(n)^{-1}\mathrm{Chord}_n(\mathbb{Z}).$$

This improves the estimate of [15] (where the analog of $b_p(n)$ is quadratic in n).

In Appendix, we prove a denominator estimate for the elements $(1, \psi)$ of the Grothendieck–Teichmüller Lie algebra $\mathfrak{gt}(\mathbb{Q}_p)$.

2 The Grothendieck–Teichmüller group

2.1 Groups GT and GRT

Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathfrak{lie}_n(\mathbb{K}) = \mathfrak{lie}(x_1, \ldots, x_n; \mathbb{K})$ be the degree completion of the free Lie algebra over $\mathbb{K}$ with generators $x_1, \ldots, x_n$. For instance, the Campbell–Hausdorff series $\mathrm{ch}(x_1, \ldots, x_n) = \ln(e^{x_1} \ldots e^{x_n})$ is an element of $\mathfrak{lie}_n(\mathbb{K})$.

Let F_n be a free group with generators $X_1, \ldots, X_n$, and PB_n be the pure braid group for n strands with standard generators $X_{i,j}$ for $i < j$ (the strand i makes a tour around the strand j). Denote by $F_n(\mathbb{K})$ and $\mathrm{PB}_n(\mathbb{K})$ their $\mathbb{K}$-prounipotent completions. By putting $x_i = \ln(X_i)$, one recovers the free Lie algebra $\mathfrak{lie}_n(\mathbb{K})$ with generators $x_1, \ldots, x_n$. Similarly, by putting $x_{i,j} = \ln(X_{i,j})$, one obtains the (filtered) Lie algebra $\mathfrak{pb}_n(\mathbb{K})$ with generators $x_{i,j}$ for $1 \le i < j \le n$. The associated graded Lie algebra is the Lie algebra $\mathfrak{t}_n(\mathbb{K})$ of infinitesimal braids with generators $t_{i,j} = t_{j,i}$ for $i, j = 1, \ldots, n$, $i \neq j$ and relations $[t_{i,j}, t_{i,k} + t_{j,k}] = 0$ for all triples i, j, k and $[t_{i,j}, t_{k,l}] = 0$ for distinct i, j, k, and l.

For a commutative ring R, we will denote by $R\langle x_1, \ldots, x_n\rangle^k$ the R-module spanned by homogeneous non-commutative polynomials of degree k with coefficients in R, by $R\langle x_1, \ldots, x_n\rangle^{\le k}$ the module spanned by non-commutative polynomials of degree at most k, and by $R\langle\langle x_1, \ldots, x_n\rangle\rangle^{\ge k}$ the module spanned by non-commutative formal power series of degree at least k. Recall that one can view $F_n(\mathbb{K})$ as the set of group-like elements in $\mathbb{K}\langle\langle x_1, \ldots, x_n\rangle\rangle$ equipped with the standard co-product $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$.

The Grothendieck–Teichmüller group $\mathrm{GT}(\mathbb{K})$ is defined in [6] as the set of pairs (λ, f) with $\lambda \in \mathbb{K}^*$ and $f \in F_2(\mathbb{K})$ such that

$$f(x, y) = f(y, x)^{-1}, \tag{2}$$
$$f(z, x)e^{mz} f(y, z)e^{my} f(x, y)e^{mx} = 1, \tag{3}$$

where $e^x e^y e^z = 1$ and $m = (\lambda - 1)/2$, and

$$\begin{aligned} &f(x_{1,2}, \mathrm{ch}(x_{2,3}, x_{2,4}))f(\mathrm{ch}(x_{1,3}, x_{2,3}), x_{3,4}) \\ &\quad = f(x_{2,3}, x_{3,4})f(\mathrm{ch}(x_{1,2}, x_{1,3}), \mathrm{ch}(x_{2,4}, x_{3,4}))f(x_{1,2}, x_{2,3}). \end{aligned} \tag{4}$$

The last equation is understood as an equality in $\mathrm{PB}_4(\mathbb{K})$. The group law of $\mathrm{GT}(\mathbb{K})$ is defined by the formula $(\lambda_1, f_1) \cdot (\lambda_2, f_2) = (\lambda, f)$, where $\lambda = \lambda_1\lambda_2$ and

$$f(x, y) = f_1(\lambda_2 f_2(x, y)x f_2(x, y)^{-1}, \lambda_2 y) f_2(x, y). \tag{5}$$

We denote by $\chi : \mathrm{GT}(\mathbb{K}) \to \mathbb{K}^*$ the group homomorphism defined by the formula $\chi(\lambda, f) = \lambda$.

The Lie algebra $\mathfrak{gt}(\mathbb{K})$ (corresponding to the group $\mathrm{GT}(\mathbb{K})$) is the set of pairs (s, ψ) with $s \in \mathbb{K}$ and $\psi \in \mathfrak{lie}_2(\mathbb{K})$ such that

$$\psi(x, y) = -\psi(y, x), \tag{6}$$
$$\psi(x, y) + \psi(y, z) + \psi(z, x) + \frac{s}{2}(x + y + z) = 0, \tag{7}$$

for $\mathrm{ch}(x, y, z) = 0$ (i.e., $z = -\mathrm{ch}(x, y)$), and

$$\begin{aligned} &\psi(x_{1,2}, \mathrm{ch}(x_{2,3}, x_{2,4})) + \psi(\mathrm{ch}(x_{1,3}, x_{2,3}), x_{3,4}) \\ &\quad = \psi(x_{2,3}, x_{3,4}) + \psi(\mathrm{ch}(x_{1,2}, x_{1,3}), \mathrm{ch}(x_{2,4}, x_{3,4})) + \psi(x_{1,2}, x_{2,3}). \end{aligned} \tag{8}$$

The last equation is understood as an equality in the Lie algebra $\mathfrak{pb}_4(\mathbb{K})$.

The kernel of the Lie homomorphism $\chi : (s, \psi) \mapsto s \in \mathbb{K}$ is the Lie subalgebra $\mathfrak{gt}_1(\mathbb{K})$. It admits a graded version $\mathfrak{grt}(\mathbb{K})$ formed by elements $\psi \in \mathfrak{lie}_2(\mathbb{K})$ satisfying Eq. (6), equation

$$\psi(x, y) + \psi(y, z) + \psi(z, x) = 0, \tag{9}$$

for $x + y + z = 0$, and

$$\begin{aligned} &\psi(t_{1,2}, t_{2,3} + t_{2,4}) + \psi(t_{1,3} + t_{2,3}, t_{3,4}) \\ &\quad = \psi(t_{2,3}, t_{3,4}) + \psi(t_{1,2} + t_{1,3}, t_{2,4} + t_{3,4}) + \psi(t_{1,2}, t_{2,3}). \end{aligned} \tag{10}$$

Here, the grading is induced by the natural grading of $\mathfrak{lie}_2(\mathbb{K})$. The corresponding group is denoted by $\mathrm{GRT}(\mathbb{K})$. This group is equipped with the group law $f_1 \cdot f_2 = f$, where $f(x, y) = f_1(f_2(x, y)\, x\, f_2(x, y)^{-1}, y) f_2(x, y)$.

2.2 The group GT over $\mathbb{Q}_p$

Let $p > 2$ be a prime, $\mathbb{Q}_p$ be the field of rational p-adic numbers, and $\mathbb{Z}_p$ be the ring of p-adic integers. Consider the subgroup $\mathrm{GT}_p \subset \mathrm{GT}(\mathbb{Q}_p)$ that consists of the pairs (λ, f) with $\lambda \in \mathbb{Z}_p^*$ and $f \in (F_2)_p$, where $(F_2)_p$ is the pro-p completion of the free group F_2. Recall [14] that the elements of $(F_2)_p$ can be viewed as group-like elements in $\mathbb{Z}_p\langle\langle\hat{x}, \hat{y}\rangle\rangle \subset \mathbb{Q}_p\langle\langle x, y\rangle\rangle$, where $\hat{x} = e^x - 1$, $\hat{y} = e^y - 1$, and the standard co-product is given on generators by $\Delta(\hat{x}) = 1 \otimes \hat{x} + \hat{x} \otimes 1 + \hat{x} \otimes \hat{x}$, $\Delta(\hat{y}) = 1 \otimes \hat{y} + \hat{y} \otimes 1 + \hat{y} \otimes \hat{y}$.

We shall need the following classical result, which follows from surjectivity of the p-adic cyclotomic character $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^*$ and from the fact that this character is the composition of a group morphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GT}_p$ with the character $\chi : \mathrm{GT}_p \to \mathbb{Z}_p^*$ (see the proof of Proposition 5.3 in [6]):

Theorem 2.1 *The character $\chi : GT_p \to \mathbb{Z}_p^*$ is surjective.*

3 *p*-adic associators

3.1 Drinfeld associators

For $\mu \in \mathbb{K}$, the set of Drinfeld associators $\mathrm{Assoc}_\mu(\mathbb{K})$ is defined as the set of group-like elements $\Phi \in \mathbb{K}\langle\langle x, y\rangle\rangle$ such that

$$\Phi(x, y) = \Phi(y, x)^{-1}, \tag{11}$$

$$e^{\mu x/2}\Phi(z, x)e^{\mu z/2}\Phi(y, z)e^{\mu y/2}\Phi(x, y) = 1 \tag{12}$$

for $x + y + z = 0$, and

$$\begin{aligned}&\Phi(t_{1,2}, t_{2,3} + t_{2,4})\,\Phi(t_{1,3} + t_{2,3}, t_{3,4})\\ &\quad = \Phi(t_{2,3}, t_{3,4})\,\Phi(t_{1,2} + t_{1,3}, t_{2,4} + t_{3,4})\,\Phi(t_{1,2}, t_{2,3})\end{aligned} \tag{13}$$

in the group $\exp(\mathfrak{t}_4)$.

Remark 3.1 It has recently been shown by Furusho [9] that Eqs. (11) and (12) are implied by the pentagon Eq. (13) and by $\Phi \in 1 + \mathbb{K}\langle\langle x, y\rangle\rangle^{\geq 2}$. The coefficient μ in (12) is determined by the expansion $\Phi(x, y) = 1 + \mu^2/24\,[x, y] + \cdots$, where the dots stand for terms of degree higher than two.

Let

$$\mathrm{Assoc}(\mathbb{K}) = \{(\mu, \Phi);\, \mu \in \mathbb{K}^*, \Phi \in \mathrm{Assoc}_\mu(\mathbb{K})\}$$

be the set of all associators with $\mu \neq 0$ ($\mathrm{Assoc}_0(\mathbb{K})$ coincides with $\mathrm{GRT}(\mathbb{K})$). The group $\mathrm{GT}(\mathbb{K})$ acts on $\mathrm{Assoc}(\mathbb{K})$ via

$$(\lambda, f) \cdot (\mu, \Phi) = (\lambda\mu,\, f(\mu\,\Phi(x, y)\,x\,\Phi(x, y)^{-1},\, \mu\,y)\,\Phi(x, y)).$$

This action is free and transitive.

The group $\mathbb{K}^*$ acts on $\mathrm{Assoc}(\mathbb{K})$ by $\lambda \cdot (\mu, \Phi) = (\lambda\mu, \Phi(\lambda x, \lambda y))$. By identifying the quotient $\mathrm{Assoc}(\mathbb{K})/\mathbb{K}^*$ with $\mathrm{Assoc}_1(\mathbb{K})$, we get an action of $\mathrm{GT}(\mathbb{K})$ on $\mathrm{Assoc}_1(\mathbb{K})$. Under this action, every $\Phi \in \mathrm{Assoc}_1(\mathbb{K})$ has a one-parameter stabilizer of the form $\{(\lambda, f_\lambda) \in \mathrm{GT}, \lambda \in \mathbb{K}^*\}$, i.e.,

$$f_\lambda(\Phi(x, y)x\Phi(x, y)^{-1}, y)\,\Phi(x, y) = \Phi(\lambda x, \lambda y). \tag{14}$$

This one-parameter subgroup of $\mathrm{GT}(\mathbb{K})$ is generated by an element $(1, \psi) \in \mathfrak{gt}(\mathbb{K})$ satisfying

$$\Phi(x, y)^{-1}\,\frac{d}{d\lambda}\Phi(\lambda x, \lambda y)|_{\lambda=1} = \psi(x, \Phi(x, y)^{-1} y\,\Phi(x, y)). \tag{15}$$

Equation (15) gives a bijective correspondence between the elements ψ of $\mathbb{K}\langle\langle x, y\rangle\rangle^{\geq 1}$ and the elements Φ of $1 + \mathbb{K}\langle\langle x, y\rangle\rangle^{\geq 1}$. By Proposition 5.2 in [6], this correspondence restricts to a bijection between $\mathrm{Assoc}_1(\mathbb{K})$ and the set of elements $(s, \psi) \in \mathfrak{gt}(\mathbb{K})$ with $s = 1$.

Theorem 3.1 *Let* $(\lambda_0, f) \in \mathrm{GT}(\mathbb{K})$ *be such that* $\lambda_0 \in \mathbb{K}^*$ *is not a root of unity. Then, there is a unique associator* $\Phi \in \mathrm{Assoc}_1(\mathbb{K})$ *such that*

$$(\lambda_0, f) \cdot (1, \Phi) = \lambda_0 \cdot (1, \Phi),$$

i.e. such that

$$f(\Phi(x, y)x\Phi(x, y)^{-1}, y)\,\Phi(x, y) = \Phi(\lambda_0 x, \lambda_0 y). \tag{16}$$

Proof Since λ_0 is not a root of 1, there is a unique element $\Phi \in 1 + \mathbb{K}\langle\langle x, y\rangle\rangle^{\geq 1}$ satisfying (16). Indeed, the degree n homogeneous part of (16) is of the form

$$\Phi_n + Q_n(x, y, \Phi_1, \ldots, \Phi_{n-1}) = \lambda_0^n \Phi_n,$$

where Q_n is a non-commutative polynomial and Φ_n is the degree n homogeneous part of Φ. We thus have a recurrence relation

$$\Phi_n = \frac{1}{\lambda_0^n - 1}\,Q_n(x, y, \Phi_1, \ldots, \Phi_{n-1}),$$

which admits a unique solution.

We need to prove that $\Phi \in \mathrm{Assoc}_1(\mathbb{K})$. Notice that for each $\lambda \in \mathbb{K}$, there is a unique $f_\lambda \in \mathbb{K}\langle\langle x, y\rangle\rangle$ satisfying (14), i.e. such that $(\lambda, f_\lambda) \cdot (1, \Phi) = \lambda \cdot (1, \Phi)$. Moreover, the degree n part of f_λ is a polynomial in λ of degree at most n. By the uniqueness property, we have $(\lambda, f_\lambda)|_{\lambda=\lambda_0} = (\lambda_0, f)$ and more generally $(\lambda, f_\lambda)|_{\lambda=\lambda_0^n} = (\lambda_0, f)^n$ for every $n \in \mathbb{Z}$. Since $(\lambda, f_\lambda) \in \mathrm{GT}(\mathbb{K})$ for infinitely many values of λ (namely for $\lambda = \lambda_0^n$), we have $(\lambda, f_\lambda) \in \mathrm{GT}(\mathbb{K})$ for every non-zero value of λ.

Define $h_\epsilon(x, y) = f_{1/\epsilon}(\epsilon x, \epsilon y)$, its coefficients are polynomials in ϵ. By making a substitution $\lambda = 1/\epsilon$ and $x \mapsto \epsilon x$, $y \mapsto \epsilon y$ in Equation (14), we get

$$\Phi(x, y) = h_\epsilon(\Phi(\epsilon x, \epsilon y)x\Phi(\epsilon x, \epsilon y)^{-1}, y)\Phi(\epsilon x, \epsilon y),$$

and hence $\Phi(x, y) = h_0(x, y)$. Relations (11), (12), (13) for Φ then follow from relations (2), (3), (4) for (λ, f_λ). Therefore, $\Phi \in \mathrm{Assoc}_1(\mathbb{K})$. $\square$

The group $\mathrm{GRT}(\mathbb{K})$ acts on $\mathrm{Assoc}_1(\mathbb{K})$ by

$$g \cdot \Phi = \Phi(g(x, y)\, x\, g(x, y)^{-1}, y)\, g(x, y).$$

This action is free and transitive.

Following Drinfled, we introduce the set of associators $\mathrm{Assoc}_1^{(k)}(\mathbb{K}) \subset \mathbb{K}\langle x, y\rangle^{\leq k} \cong \mathbb{K}\langle\langle x, y\rangle\rangle/\mathbb{K}\langle\langle x, y\rangle\rangle^{\geq k+1}$ satisfying Eqs. (11), (12) (with $\mu = 1$), and (13) up to degree k. By Proposition 5.8 in [6], the natural projections $\mathrm{Assoc}_1^{(k+1)}(\mathbb{K}) \to \mathrm{Assoc}_1^{(k)}(\mathbb{K})$ are surjective for all k. One can also introduce Lie algebras $\mathfrak{gt}^{(k)}(\mathbb{K}) \subset \mathbb{K}\langle x, y\rangle^{\leq k}$, which consist of the pairs (s, ψ) with $s \in \mathbb{K}$, $\psi \in \mathfrak{lie}_2(\mathbb{K}) \cap \mathbb{K}\langle x, y\rangle^{\leq k}$ satisfying Eqs. (6), (7) and (8) up to degree k. We will need the following version of Proposition 5.2 in [6].

Proposition 3.2 *For all k, formula* (15) *gives a one-to-one correspondence between the elements* $(1, \psi) \in \mathfrak{gt}^{(k)}(\mathbb{K})$ *and the elements* $\Phi \in \mathrm{Assoc}_1^{(k)}(\mathbb{K})$.

Proof Let $(1, \psi) \in \mathfrak{gt}^{(k)}(\mathbb{K})$. By Proposition 5.6 in [6], the Lie algebra $\mathfrak{gt}(\mathbb{K})$ is isomorphic (in a non-canonical way) to the semi-direct sum $\mathbb{K} \oplus \mathfrak{grt}(\mathbb{K})$. We choose such an isomorphism $\tau : \mathfrak{gt}(\mathbb{K}) \to \mathbb{K} \oplus \mathfrak{grt}(\mathbb{K})$, and apply it to $(1, \psi)$ to obtain an element $1 + \phi \in \mathbb{K} \oplus \mathfrak{grt}^{(k)}(\mathbb{K})$. Since $\mathfrak{grt}(\mathbb{K})$ is graded, the element $1 + \phi$ lifts (in many ways) to an element $1 + \hat{\phi} \in \mathbb{K} \oplus \mathfrak{grt}(\mathbb{K})$. By applying the inverse of τ, we arrive at $(1, \hat{\psi}) \in \mathfrak{gt}(\mathbb{K})$, which is a lift of $(1, \psi)$. Consider the associator $\hat{\Phi}$ corresponding to the element $(1, \hat{\psi})$, and let Φ be its degree k truncation. Equation (15) is verified for $\hat{\psi}$ and $\hat{\Phi}$. By truncating it at degree k, we can replace $\hat{\psi}$ by ψ and $\hat{\Phi}$ by Φ.

Similarly, let $\Phi \in \mathrm{Assoc}_1^{(k)}(\mathbb{K})$, and let $\hat{\Phi} \in \mathrm{Assoc}_1(\mathbb{K})$ be an associator lifting Φ. We define $\hat{\psi}$ by Eq. (15) to obtain $(1, \hat{\psi}) \in \mathfrak{gt}(\mathbb{K})$, and let ψ be the degree k truncation of $\hat{\psi}$. Again, by truncating Eq. (15) at degree k, we can replace $\hat{\psi}$ by ψ and $\hat{\Phi}$ by Φ. $\square$

3.2 Denominator estimates of associators over $\mathbb{Q}_p$

In this section, we prove existence of p-adic associators with certain denominator bounds. The bounds are given by a function a_p defined as

$$a_p(n) = \left[\frac{n}{p-1}\right] + v_p\left(\left[\frac{n}{p-1}\right]!\right), \tag{17}$$

where v_p is the p-adic valuation.

Proposition 3.3 *The function $a_p(n)$ has the following properties*

$$a_p(n) \geq a_p(k) + a_p(l) \text{ if } n \geq k + l, \tag{18}$$
$$a_p(m+n) \geq a_p(m+1) + v_p(n!), \tag{19}$$

Proof Inequality (18) follows from $v_p\left((k+l)!\right) \geq v_p(k!) + v_p(l!)$ and inequality (19) from $v_p(n!) \leq [(n-1)/(p-1)]$. □

Property (18) ensures that $\sum_{n=0}^{\infty} p^{-a_p(n)}\mathbb{Z}_p\langle x, y\rangle^n$ is a subring of $\mathbb{Q}_p\langle\langle x, y\rangle\rangle$, and property (19) implies

$$\sum_{n=0}^{\infty} p^{-a_p(n)}\mathbb{Z}_p\langle x, y\rangle^n = \sum_{n=0}^{\infty} p^{-a_p(n)}\mathbb{Z}_p\langle \hat{x}, \hat{y}\rangle^n,$$

for $\hat{x} = e^x - 1$, and $\hat{y} = e^y - 1$.

For any prime $p > 2$, we can choose an element $\lambda \in \mathbb{Z}_p^*$ generating a dense subgroup of $\mathbb{Z}_p^*$, i.e., such that $\lambda \mod p$ is a generator of $\mathbb{F}_p^*$ and $\lambda^{p-1} \in 1 + p\mathbb{Z}_p^*$.

Theorem 3.4 *Let p be a prime. For $p > 2$, let $(\lambda, f) \in GT_p$ be such that λ generates a dense subgroup of $\mathbb{Z}_p^*$, and, for $p = 2$, let $(\lambda, f) \in GT_2$ be such that $\lambda \in 1+4+8\mathbb{Z}_2$. Then, there is a unique Drinfeld associator $\Phi \in \mathrm{Assoc}_1(\mathbb{Q}_p)$ such that*

$$f(\Phi(x, y)\, x\, \Phi(x, y)^{-1}, y)\, \Phi(x, y) = \Phi(\lambda x, \lambda y). \tag{20}$$

The p-adic associator Φ satisfies the denominator estimate

$$\Phi \in \sum_{n=0}^{\infty} p^{-a_p(n)}\mathbb{Z}_p\langle x, y\rangle^n. \tag{21}$$

Proof Existence and uniqueness of an associator $\Phi \in \mathrm{Assoc}_1(\mathbb{Q}_p)$ satisfying Equation (20) follow from Theorem 3.1.

Let $\hat{\Phi} \in \mathbb{Q}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle$ be the formal power series in $\hat{x}$ and $\hat{y}$, which coincides with $\Phi \in \mathbb{Q}_p\langle\langle x, y\rangle\rangle$ under the substitution $\hat{x} = e^x - 1$, $\hat{y} = e^y - 1$. We will prove that $\hat{\Phi} \in \sum_{n=0}^{\infty} p^{-a_p(n)}\mathbb{Z}_p\langle \hat{x}, \hat{y}\rangle^n$. By induction, we assume that $\hat{\Phi}$ has this property up to degree $n-1$. Then, the degree n part of Eq. (20) reads

$$\lambda^n \hat{\Phi}_n(\hat{x}, \hat{y}) = \hat{\Phi}_n(\hat{x}, \hat{y}) + F_n(\hat{x}, \hat{y}, \hat{\Phi}_1, \ldots, \hat{\Phi}_{n-1}),$$

where $F_n(\hat{x}, \hat{y}, \hat{\Phi}_1, \ldots, \hat{\Phi}_{n-1})$ is a non-commutative polynomial in $\hat{x}$, $\hat{y}$ and $\hat{\Phi}_k(\hat{x}, \hat{y})$ for $k < n$ with coefficients in $\mathbb{Z}_p$. Solving for $\hat{\Phi}_n$, we obtain

$$\hat{\Phi}_n = \frac{1}{\lambda^n - 1} F_n(\hat{x}, \hat{y}, \hat{\Phi}_1, \ldots, \hat{\Phi}_{n-1}).$$

For $p > 2$, we note that $(\lambda^n - 1)^{-1} \in \mathbb{Z}_p$ if $(p-1) \nmid n$ and that $(\lambda^{(p-1)k} - 1)^{-1} \in (kp)^{-1}\mathbb{Z}_p$. Since F_n satisfies $F_n(0, 0, \hat{\Phi}_1, \ldots, \hat{\Phi}_{n-1}) = 0$, we use (18) to obtain

$$F_n(\hat{x}, \hat{y}, \hat{\Phi}_1, \ldots, \hat{\Phi}_{n-1}) \in p^{-a_p(n-1)}\mathbb{Z}_p\langle \hat{x}, \hat{y}\rangle^n$$

and thus

$$\hat{\Phi}_n \in \left\{ \begin{array}{ll} p^{-a_p(n-1)}\mathbb{Z}_p\langle \hat{x}, \hat{y}\rangle^n & \text{if } (p-1) \nmid n \\ p^{-a_p(n-1)-1-v_p(n/(p-1))}\mathbb{Z}_p\langle \hat{x}, \hat{y}\rangle^n & \text{if } (p-1) \mid n \end{array} \right\} = p^{-a_p(n)}\mathbb{Z}_p\langle \hat{x}, \hat{y}\rangle^n.$$

For $p = 2$, we have $(\lambda^n - 1)^{-1} \in (4n)^{-1}(1 + 2\mathbb{Z}_2)$. In this case, the induction argument gives

$$\hat{\Phi}_n \in n^{-1}2^{-a_2(n-2)-2}\mathbb{Z}_2\langle \hat{x}, \hat{y}\rangle^n.$$

Here, we used the fact that, for $(\lambda, f) \in \mathrm{GT}$, the degree one component of f always vanishes. Since

$$a_2(n) \geq v_2(n) + a_2(n-2) + 2\,,$$

we have

$$\hat{\Phi}_n \in 2^{-a_2(n)}\mathbb{Z}_2\langle \hat{x}, \hat{y}\rangle^n,$$

as required. □

Remark 3.2 A p-adic associator Φ satisfying condition (21) is convergent in the p-adic norm as a power series in x and y if $v_p(x), v_p(y) > p/(p-1)^2$; under this condition, we have $v_p(\Phi - 1) > 1/(p-1)$. This follows from the inequality

$$a_p(n) \leq \frac{p}{(p-1)^2}\, n - \frac{1}{p-1}.$$

Remark 3.3 Let $(\lambda, f) \in \mathrm{GT}_p$ be as in Theorem 3.4, and let $(0, g)$ be the unique element of the Grothendieck–Teichmüller semi-group defined by the property $(0, g)(\lambda, f) = (0, g)$. Again, one can show that $g \in \sum_{n=0}^{\infty} p^{-a_p(n)}\mathbb{Z}_p\langle x, y\rangle^n$.

Remark 3.4 In [10], Furusho defined a p-adic analog Φ^p_{KZ} of the KZ associator, which turns out to be an element of $\mathrm{GRT}(\mathbb{C}_p)$ rather than of $\mathrm{Assoc}_1(\mathbb{Q}_p)$. Hence, one cannot directly apply the considerations of this paper to this element.

3.3 Kashiwara–Vergne factorization of the Jacobson element

It is natural to ask whether the denominator estimate given by Theorem 3.4 is optimal. Despite the fact that we are not able to answer this question in full generality, we will show in this section that the simple pole (i.e., p^{-1}) must appear in degree $p-1$ as suggested by our estimate.

Let $\Phi \in \mathrm{Assoc}_1(\mathbb{Q}_p)$ be such that

$$\Phi \in 1 + \mathbb{Z}_p\langle\langle x, y\rangle\rangle^{\geq 1} + p^{-1}\mathbb{Z}_p\langle x, y\rangle^{p-1} + \mathbb{Q}_p\langle\langle x, y\rangle\rangle^{\geq p}.$$

For instance, the Φ's constructed in Theorem 3.4 are all of this type. Let $(1, \psi) \in \mathfrak{gt}(\mathbb{Q}_p)$ be given by relation (15). It then satisfies

$$\psi \in \mathbb{Z}_p\langle\langle x, y\rangle\rangle^{\geq 1} + p^{-1}\mathbb{Z}_p\langle x, y\rangle^{p-1} + \mathbb{Q}_p\langle\langle x, y\rangle\rangle^{\geq p}.$$

Denote by $\sigma_p \in \mathfrak{lie}_2(\mathbb{F}_p)$ the modulo p reduction of the degree $p-1$ part of $p\,\psi$. Recall that in the free Lie algebra $\mathfrak{lie}_2(\mathbb{F}_p)$ there is a canonical (Jacobson) element given by the formula $J_p(x, y) = (x+y)^p - x^p - y^p$.

Proposition 3.5 *The element σ_p satisfies Eqs.* (6)*,* (9)*, and* (10) *and the property*

$$[x, \sigma_p(-x-y, x)] + [y, \sigma_p(-x-y, y)] = J_p(x, y), \tag{22}$$

Proof By taking the degree $p-1$ parts of Eqs. (6) and (7), and extracting the residues (i.e., the coefficients in front of p^{-1}), we obtain the following equations for σ_p:

$$\sigma_p(x, y) = -\sigma_p(y, x)\ , \quad \sigma_p(x, y) + \sigma_p(y, z) + \sigma_p(z, x) = 0$$

for $x + y + z = 0$. Here, we used the fact that the degree $p-1$ part of $\mathrm{ch}(x, y)$ has coefficients in $\mathbb{Z}_p$. By applying the same procedure to Eq. (8), we obtain

$$\begin{aligned}\sigma_p(x_{1,2}, x_{2,3} + x_{2,4}) + \sigma_p(x_{1,3} + x_{2,3}, x_{3,4})&\\ = \sigma_p(x_{2,3}, x_{3,4}) + \sigma_p(x_{1,2} + x_{1,3}, x_{2,4} + x_{3,4}) &+ \sigma_p(x_{1,2}, x_{2,3}),\end{aligned}$$

where relations of the Lie algebra $\mathfrak{pb}_4(\mathbb{Q}_p)$ are replaced by their linearized (graded) version in the Lie algebra $\mathfrak{t}_4(\mathbb{F}_p)$.

For Eq. (22), we add up two copies of Eq. (7), one for the triple $x, y, z = -ch(x, y)$ and one for the tripe $y, x, \tilde{z} = -\mathrm{ch}(y, x)$, to obtain

$$(1 - \exp(-\mathrm{ad}_x))\psi(z, x) + (\exp(\mathrm{ad}_y) - 1)\psi(z, y) + x + y + \frac{1}{2}(z + \tilde{z}) = 0.$$

We consider the residue of the degree p component of this equation and use the fact that the residue of the degree p part of $\mathrm{ch}(x, y)$ is exactly $J_p(x, y)$. This gives (22), as required. $\square$

Proposition 3.5 shows that $\sigma_p(x, y)$ does not vanish, and ψ has to have a simple pole in degree $p-1$. For Φ_{p-1}, the degree $p-1$ homogeneous component of Φ, relation (15) implies $\sigma_p = -p\,\Phi_{p-1} \bmod p$. Therefore, Φ has a simple pole in degree $p-1$.

Every element $a \in \mathbb{K}\langle x, y\rangle$ admits a unique decomposition of the form $a = a_0 + a_1 x + a_2 y$, where $a_0 \in \mathbb{K}$ and $a_1, a_2 \in \mathbb{K}\langle x, y\rangle$. We will denote $a_1 = \partial_x a$, $a_2 = \partial_y a$. Let $\mathfrak{tr}_2(\mathbb{K})$ be the quotient of $\mathbb{K}\langle x, y\rangle$ by the subspace spanned by commutators. The space $\mathfrak{tr}_2(\mathbb{K})$ is spanned by cyclic words in letters x and y. We will denote by $\mathrm{tr} : \mathbb{K}\langle x, y\rangle \to \mathfrak{tr}_2(\mathbb{K})$ the canonical projection.

Proposition 3.6 *Let $a(x, y) = \sigma_p(-x - y, x)$ and $b(x, y) = \sigma_p(-x - y, y)$. Then,*

$$\mathrm{tr}(x(\partial_x a) + y(\partial_y b)) = \frac{1}{2}\mathrm{tr}\left((x + y)^{p-1} - x^{p-1} - y^{p-1}\right). \tag{23}$$

Proof By Propositions 4.2 and 4.3 in [2], Eqs. (6), (9), and (10) for $\sigma_p(x, y)$ imply that $\mathrm{tr}(x(\partial_x a) + y(\partial_y b)) = \mathrm{tr}(f(x + y) - f(x) - f(y))$, where f is a polynomial in one variable. Since the degree of the left-hand side is equal to $p - 1$, we conclude that $f(x) = \alpha x^{p-1}$ for some $\alpha \in \mathbb{F}_p$.

In order to determine α, let $a(x, y) = u\, \mathrm{ad}_x^{p-2}\, y + \cdots$, $b(x, y) = v\, \mathrm{ad}_x^{p-2}\, y + \cdots$, where the dots stand for the lower degree terms in x. By analyzing the terms of the form $x^{p-1}y$ in Eq. (22), we find $u = -1$. By looking at the terms of the form $yx^{p-2}y$, we obtain $v = 1/2$. This yields $\mathrm{tr}(x(\partial_x a) + y(\partial_y b)) = -1/2\, \mathrm{tr}(x^{p-2}y) + \cdots$. Hence, $f(x) = x^{p-1}/2$, as required. □

Remark 3.5 Equations (22) and (23) define the Kashiwara–Vergne type factorization problem for the Jacobson element. It was posed by Duflo [7]. A partial solution was given by the second author in [17]. Propositions 3.5 and 3.6 settle the problem in the positive in full generality.

4 Rational associators

4.1 The group $\mathrm{GRT}_{\mathrm{nat}}$

Let p be a prime. Following [14], we extend the valuation v_p from $\mathbb{Q}$ to $\mathbb{Q}\langle\langle x, y\rangle\rangle$ by setting $v_p(x) = v_p(y) = p/(p-1)^2$, and consider a subset of $\mathfrak{grt}(\mathbb{Q})$

$$\mathfrak{grt}_{nat} = \{\psi \in \mathfrak{grt}(\mathbb{Q});\ v_p(\psi) \geq 1/(p-1) \text{ for all } p \text{ prime}\}.$$

In other words, for $\psi(x, y) \in \mathfrak{grt}_{nat}$, we have $\psi \in \sum_{n=1}^{\infty} p^{-b(n)}\mathbb{Z}_p\langle x, y\rangle$, where

$$b(n) = \left[\frac{p}{(p-1)^2}\, n - \frac{1}{p-1}\right].$$

It is easy to see that $\mathfrak{grt}_{nat}$ is a Lie subalgebra over $\mathbb{Z}$ of $\mathfrak{grt}(\mathbb{Q})$. Indeed, $v_p([\phi, \psi]_{\mathfrak{grt}}) \geq v_p(\phi) + v_p(\psi) \geq 2/(p-1)$.

Similarly, we introduce

$$\mathrm{GRT}_{nat} = \{g \in \mathrm{GRT}(\mathbb{Q});\ v_p(g - 1) \geq 1/(p-1) \text{ for all } p \text{ prime}\}.$$

Again, for $g \in \mathrm{GRT}_{nat}$, we have $g(x, y) \in 1 + \sum_{n=1}^{\infty} p^{-b(n)} \mathbb{Z}_p\langle x, y\rangle$. GRT_{nat} is a subgroup of $\mathrm{GRT}(\mathbb{Q})$.

Proposition 4.1 *The exponential map* $\exp_{\mathfrak{grt}} : \mathfrak{grt}_{nat} \to \mathrm{GRT}_{nat}$ *is a bijection.*

Proof The $\mathfrak{grt}$ exponential map $\exp_{\mathfrak{grt}} : \psi \mapsto g$ is described by the formula $g = \sum_{n=0}^{\infty} g_n/n!$, where $g_0 = 1$ and $g_n = \psi \cdot g_{n-1} + g_{n-1}\psi$. In particular, $g_1 = \psi$ and $v_p(g_1) \geq 1/(p-1)$. By induction, we obtain $v_p(g_n) \geq n/(p-1)$, and so $v_p(g_n/n!) = v_p(g_n) - v_p(n!) \geq 1/(p-1)$, where we used the estimate $v_p(n!) \leq (n-1)/(p-1)$. Hence, $v_p(g-1) \geq 1/(p-1)$, as required.

The GRT logarithmic map $\ln_{\mathrm{GRT}} : g \mapsto \psi$ is given by $\psi = \sum_{n=1}^{\infty}(-1)^{n+1}\psi_n/n$, where $\psi_1 = g - 1$ and

$$\psi_n(x, y) = \psi_{n-1}(gxg^{-1}, y) - \psi_{n-1}(x, y) + \psi_{n-1}(gxg^{-1}, y)\,\psi_1(x, y).$$

Using $v_p(\psi_1) = v_p(\mathfrak{g} - 1) \geq 1/(p-1)$, we prove by induction that $v_p(\psi_n) \geq n/(p-1)$. Therefore, $v_p(\psi_n/n) \geq n/(p-1) - v_p(n) \geq n/(p-1) - v_p(n!) \geq 1/(p-1)$, and $v_p(\psi) \geq 1/(p-1)$ as required. □

Remark 4.1 The inequality $v_p(k!) \leq (k-1)/(p-1)$ implies

$$a_p(n) \leq b_p(n).$$

4.2 Natural rational associators

We call a rational associator $\Phi \in \mathrm{Assoc}_1(\mathbb{Q})$ *natural* if it satisfies the denominator estimates $\Phi \in 1 + \sum_{n=1}^{\infty} p^{-b_p(n)} \mathbb{Z}_p\langle x, y\rangle^n$ for all prime p, i.e., if $v_p(\Phi - 1) \geq 1/(p-1)$ for every p. We denote by Assoc_{nat} the set of (rational) natural associators; similarly, we set $\mathrm{Assoc}_{nat}^{(k)} = \mathrm{Assoc}_1(\mathbb{Q})^{(k)} \cap \left(1 + \sum_{n=1}^{k} p^{-b_p(n)} \mathbb{Z}_p\langle x, y\rangle^n\right)$. The main result of this paper is as follows:

Theorem 4.2 *The set of natural associators* Assoc_{nat} *is non-empty, and, for every k, the truncation map* $\mathrm{Assoc}_{nat} \to \mathrm{Assoc}_{nat}^{(k)}$ *is surjective.*

Proof For each prime p, we choose an associator $\Phi_p \in \mathrm{Assoc}_1(\mathbb{Q}_p)$ satisfying denominator estimates (21). Let $(1, \psi_p)$ correspond to Φ_p via relation (15). We denote by $(1, \psi_p^{(k)})$ and $\Phi_p^{(k)}$ their degree k truncations.

We choose an element $(1, \tilde{\psi}) \in \mathfrak{gt}(\mathbb{Q})$. For each $k \in \mathbb{N}$, let $(1, \tilde{\psi}^{(k)}) \in \mathfrak{gt}^{(k)}(\mathbb{Q})$ be its degree k truncation, and let $\phi_i, i = 1, \ldots, N_k$ be an integral basis in $\mathfrak{gt}_1^{(k)}(\mathbb{Q})$ (i.e., ϕ_i's are non-commutative polynomials in x and y with integer coefficients). Let P_k be the finite set of primes that either do not exceed k or enter denominators of $\tilde{\psi}^{(k)}$. For each $p \in P_k$, we choose two sufficiently large positive integers $s_k(p), t_k(p)$ (the choice will be discussed later).

For each $p \in P_k$, we have

$$(1, \psi_p^{(k)}) = (1, \tilde{\psi}^{(k)}) + \sum_{i=1}^{N_k} \kappa_{p,i}\phi_i$$

with $\kappa_{p,i} \in \mathbb{Q}_p$. Put $M_k = \prod_{p \in P_k} p^{s_k(p)}$. If the numbers $s_k(p)$ are sufficiently large, we have $M\kappa_{p,i} \in \mathbb{Z}_p$ for all $p \in P_k$ and for all i. By Chinese Remainder Theorem, we can choose integers $\mu_i \in \mathbb{Z}$ such that $\mu_i - M\kappa_{p,i} = 0 \bmod p^{t_k(p)}$ for all $p \in P_k$. Define an element

$$(1, \psi^{(k)}) = (1, \tilde{\psi}^{(k)}) + \frac{1}{M_k} \sum_{i=1}^{N_k} \mu_i \phi_i \in \mathfrak{gt}^{(k)}(\mathbb{Q}),$$

and the corresponding (by Proposition 3.2) associator $\Phi^{(k)} \in \mathrm{Assoc}_1^{(k)}(\mathbb{Q})$.

Note that, for $p \notin P_k$, the coefficients of $\psi^{(k)}$ are in $\mathbb{Z}_p$. Since $p > k$, the associator $\Phi^{(k)}$ also has coefficients in $\mathbb{Z}_p$. Let $p \in P_k$. Since the map $\psi^{(k)} \mapsto \Phi^{(k)}$ (defined by Eq. (15)) is continuous in the p-adic topology, we can choose sufficiently large $t_k(p)$'s so that $\Phi^{(k)}$ be arbitrarily close to $\Phi_p^{(k)}$. In particular, we may assume that $\Phi^{(k)} \in \mathrm{Assoc}_{nat}^{(k)}$.

Denote by $\pi_k : \mathrm{Assoc}_1^{(k+1)}(\mathbb{Q}) \to \mathrm{Assoc}_1^{(k)}(\mathbb{Q})$ the natural projection forgetting the degree $k+1$ part of an associator. Consider the associators $\Phi^{(k)}$ and $\pi_k(\Phi^{(k+1)}) \in \mathrm{Assoc}_1^{(k)}(\mathbb{Q})$. There is a unique element $g_k \in \mathrm{GRT}^{(k)}(\mathbb{Q})$ taking one of them to the other. Let $p \notin P_k \cap P_{k+1}$. Then, both $\Phi^{(k)}$ and $\pi_k(\Phi^{(k+1)})$ have coefficients in $\mathbb{Z}_p$, and so does g_k. For $p \in P_k \cap P_{k+1}$, we choose sufficiently large $t_k(p)$ and $t_{k+1}(p)$ to obtain $g_k \in \mathrm{GRT}_{nat}^{(k)}$. By Proposition 4.1, there is an isomorphism between $\mathfrak{grt}_{nat}$ and GRT_{nat}. Since all elements of $\mathfrak{grt}_{nat}^{(k)}$ lift to elements of $\mathfrak{grt}_{nat}$, the same applies to lifting from $\mathrm{GRT}_{nat}^{(k)}$ to GRT_{nat}. Hence, g_k can be lifted (in many ways) to an element $\hat{g}_k \in \mathrm{GRT}_{nat}$.

We consider the sequence of elements

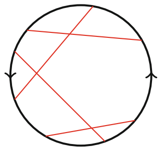
$$\Phi(k) = (\hat{g}_k \hat{g}_{k-1} \ldots \hat{g}_2) \cdot \Phi^{(k)} \in \mathrm{Assoc}_1^{(k)}(\mathbb{Q}),$$

where the action of GRT on $\mathrm{Assoc}_1^{(k)}(\mathbb{Q})$ is defined by forgetting degrees higher than k. Since GRT_{nat} and Assoc_{nat} are defined by the same denominator conditions, the action of the group GRT_{nat} preserves Assoc_{nat} and is free and transitive. We thus have $\Phi(k) \in \mathrm{Assoc}_{nat}^{(k)}$. By construction of $\hat{g}_k$, we have $\pi_k(\Phi(k+1)) = \Phi(k)$. Hence, the sequence $\Phi(k)$ defines an element of Assoc_{nat}. That is, the set Assoc_{nat} is non-empty, as required.

Finally, since the action of GRT_{nat} on Assoc_{nat} is free and transitive, any choice of an associator $\Phi \in \mathrm{Assoc}_{nat}$ gives a bijection $\mathrm{GRT}_{nat} \to \mathrm{Assoc}_{nat}$, which is moreover compatible with the truncation maps. The surjectivity of $\mathrm{Assoc}_{nat} \to \mathrm{Assoc}_{nat}^{(k)}$ therefore follows from surjectivity of $\mathrm{GRT}_{nat} \to \mathrm{GRT}_{nat}^{(k)}$. □

Remark 4.2 Let $\mathrm{Assoc}'_{nat} \subset \mathrm{Assoc}_{nat}$ be the set of rational associators satisfying the denominator estimate $\Phi \in \sum_{n=0}^{\infty} p^{-a_p(n)} \mathbb{Z}_p\langle x, y\rangle^n$. We can still prove that Assoc'_{nat} is non-empty, if we replace GRT_{nat} with a similar group given by $v_p(x) = v_p(y) = 1/(p-1)$. However, we were not able to establish the surjectivity property for truncation maps of Assoc'_{nat}.

Fig. 1 A chord diagram



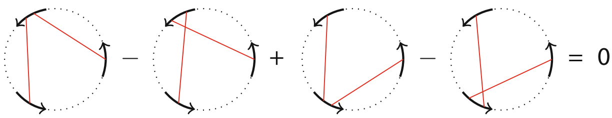
Remark 4.3 Theorem 4.2 remains valid if we require natural associators to be even (that is, $\Phi(x, y) = \Phi(-x, -y)$). Indeed, let $\Phi^{(2k)} \in \mathrm{Assoc}_1^{(2k)}(\mathbb{Q})$ be an even natural associator up to degree $2k$. By Remark 1 (after Proposition 5.8) in [6], $\Phi^{(2k)}$ can be extended by zero in degree $2k + 1$ to obtain an even natural associator up to degree $2k + 1$. Then, by Theorem 4.2, $\Phi^{(2k)}$ extends (in many ways) to a natural associator $\Phi^{(2k+2)}$ up to degree $2k + 2$. $\Phi^{(2k+2)}$ is even since $\Phi^{(2k)}$ is even and the contribution in degree $2k+1$ vanishes. Hence, even natural associators can be extended from lower to higher degrees without obstructions, as required.

Remark 4.4 In [1], it is proved that the Knizhnik–Zamolodchikov associator Φ_{KZ} has a finite convergence radius (as a non-commutative power series). It is natural to conjecture the existence of rational natural associators with finite convergence radius. We do not know how to prove (or disprove) this conjecture. More generally, it would be interesting to establish the existence of rational associators with given bounds on both denominators and numerators.

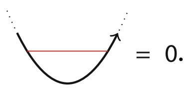
5 Kontsevich knot invariant

Drinfeld associators can be used to define a universal finite type invariant of knots in $\mathbb{R}^3$. Below we give a brief account of this construction.

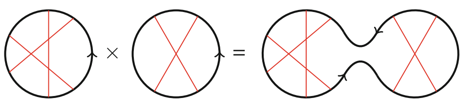
Let $\mathbb{K}$ be a field of characteristic zero. A chord diagram (see Fig. 1) is an oriented circle with a choice of pairs of marked points modulo orientation preserving diffeomorphisms. The two points of a pair are connected by a chord. The algebra $\mathrm{Chord}(\mathbb{K})$ is spanned by the chord diagrams modulo the 4T relation



and modulo the relation



The product in $\mathrm{Chord}(\mathbb{K})$ is defined as a connected sum of chord diagrams. The 4T relation insures that the product map is well-defined.



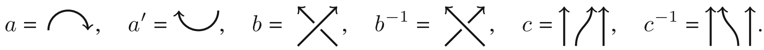
The algebra Chord($\mathbb{K}$) is graded with the grading given by the number of chords. We assume that Chord($\mathbb{K}$) is degree completed to allow infinite linear combinations of chord diagrams. The linear combinations of chord diagrams with integer coefficients span a lattice Chord($\mathbb{Z}$) $\in$ Chord($\mathbb{K}$). We define a subring (over $\mathbb{Z}$) of Chord($\mathbb{Q}$),

$$\mathrm{Chord}_{nat} = \sum_{n=0}^{\infty} D(n)^{-1}\mathrm{Chord}_n(\mathbb{Z}),$$

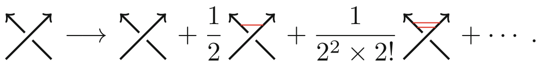
where $D(n)$ is given by Eq. (1) and $\mathrm{Chord}_n(\mathbb{Z})$ is the graded component of degree n of Chord($\mathbb{Z}$).

There are several steps in constructing the Kontsevich knot invariant (for details, see [4,16]). First, one associates to the knot K a knot diagram $\tilde{K}$ (a projection of K onto a 2-plane). By a small perturbation of the embedding, one can choose $\tilde{K}$ in such a way that it has only transversal simple crossings. One can also assume that the height (the y-coordinate) is a Morse function and that all the crossings and critical points of y have different heights.

One can then decompose the knot diagram into a sequence of basic elements. There are six possible basic elements (up to change of orientation),



Then one decorates the basic elements with (horizontal) chords according to the following rules: elements a and a' remain unchanged, the element b is decorated with $\exp(t/2)$, where t is a horizontal chord extending between two strands,



Similarly, the element b^{-1} is decorated with $\exp(-t/2)$. Finally, one chooses a Drinfeld associator $\Phi \in \mathrm{Assoc}_1(\mathbb{K})$ and decorates elements c and c^{-1} with $\Phi(t_{1,2}, t_{2,3})$ and $\Phi(t_{1,2}, t_{2,3})^{-1}$, respectively. Here, $t_{1,2}$ is a horizontal chord extending between strands 1 and 2 and $t_{2,3}$ is a horizontal chord extending between strands 2 and 3. To get a decoration of basic elements with some strands with reversed orientation one uses the following rule: let S be a strand with k marked points (ends of various chords). Then, reversing orientation of S is accompanied by adding the sign $(-1)^k$.

The decorated knot diagram is an element of Chord($\mathbb{K}$). We denote it by $I_\Phi(\tilde{K})$. By abuse of notation, we denote by $\tilde{K}$ the knot diagram together with its decomposition into basic elements.

Proposition 5.1 *Let Φ be a natural rational associator. Then, $I_\Phi(\tilde{K}) \in \mathrm{Chord}_{nat}$, for all knot diagrams $\tilde{K}$.*

Proof Assume that the knot diagram $\tilde{K}$ decomposes into s basic elements of type b and b^{-1}, t basic elements of type c and c^{-1}, and some number of basic elements of type a and a'. Then, denominators of degree n contributions in $I_\Phi(\tilde{K})$ are of the form

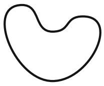
$$D(k_1)\dots D(k_s)(2^{l_1} l_1!)\dots(2^{l_t} l_t!),$$

where $k_1 + \cdots + k_s + l_1 + \cdots + l_t = n$. Here, contributions $D(k_1), \ldots, D(k_s)$ come from denominators of the natural associator Φ, and remaining terms are denominators of the exponential functions $\exp(t/2)$ and $\exp(-t/2)$.

Since $b_p(k+l) \geq b_p(k) + b_p(l)$, the product $D(k_1) \ldots D(k_s)$ is a divisor of $D(k)$ for $k = k_1 + \cdots + k_s$. Furthermore, for $p \geq 3$, we have $b_p(k+l) \geq b_p(k) + v_p(l!)$ and $b_2(k+l) \geq b_2(k) + v_2(l!) + l$. Hence, $D(k)(2^{l_1} l_1!) \ldots (2^{l_t} l_t!)$ is a divisor $D(n) = D(k+l)$ for $l = l_1 + \cdots + l_t$. □

Note that, for all $\tilde{K}$, we have $I_\Phi(\tilde{K}) \in 1 + \mathrm{Chord}_{\geq 1}(\mathbb{K})$. Hence, if Φ is a natural rational associator, $I_\Phi(\tilde{K})$ is an invertible element of Chord_{nat}.

Let $\tilde{K}_0$ be the following knot diagram:



$\tilde{K}_0$

The Kontsevich invariant of K is defined as

$$I(K) = I_\Phi(\tilde{K}) I_\Phi(\tilde{K}_0)^{-X},$$

where X is the number of local maxima of the height function on $\tilde{K}$. The information about $I(K)$ is summarized in the following theorem (see [16]).

Theorem 5.2 *$I(K)$ is a knot invariant. That is, it is independent of the choice of the knot diagram associated to the knot K, of the decomposition of $\tilde{K}$ into basic elements, and of the choice of a Drinfeld associator Φ.*

The main result of this Section is the following theorem.

Theorem 5.3 *The Kontsevich invariant $I(K) \in \mathrm{Chord}_{nat}$, for all knots K.*

Proof Let K be a knot and Φ be a natural rational associator. For any knot diagram $\tilde{K}$ of K and for any decomposition of $\tilde{K}$ into basic elements, we have $I_\Phi(\tilde{K}) \in \mathrm{Chord}_{nat}$. We also have $I_\Phi(\tilde{K}_0) \in \mathrm{Chord}_{nat}$, and $I_\Phi(\tilde{K}_0)$ is an invertible element. Hence, $I(K) \in \mathrm{Chord}_{nat}$, as required. □

Remark 5.1 A weaker estimate (the function $\tilde{b}_p(n)$ is quadratic in n) on denominators in the Kontsevich knot invariant was obtained in [15]. To get this estimate, the author uses associators with values in "Chinese characters" instead of Drinfeld associators.

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Appendix A. Denominator estimates for elements of $\mathfrak{gt}(\mathbb{Q}_p)$

Here, we prove an estimate on the elements $(1, \psi) \in \mathfrak{gt}(\mathbb{Q}_p)$ corresponding to the p-adic associators constructed in Theorem 3.4 via relation (15). In contrast to the linear estimate of Theorem 3.4, the new estimate has a logarithmic growth. However, this estimate is only satisfied by ψ viewed as a non-commutative series in $\hat{x} = e^x - 1$ and $\hat{y} = e^y - 1$.

For $p > 2$, let $(\lambda, f) \in \mathrm{GT}(\mathbb{Q}_p)$ and $\Phi \in \mathrm{Assoc}_1(\mathbb{Q}_p)$ be as in Theorem 3.4. Define $(1, \psi) = \ln(\lambda', f')/\ln \lambda' \in \mathfrak{gt}(\mathbb{Q}_p)$, where $(\lambda', f') = (\lambda, f)^{p-1}$. Then, Φ and ψ satisfy relation (15). Since $\Phi^{(p-2)}$ has coefficients in $\mathbb{Z}_p$, the same is true for $\psi^{(p-2)}$. Using the facts that $\ln \lambda' \in p\mathbb{Z}_p$, $f' \in \mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle$, and $(\lambda', f') = \exp\big(\ln(\lambda')(1, \psi)\big)$, we obtain

$$f' \in 1 + p\mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq 1} + \mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq p-1}. \tag{24}$$

Proposition A.1 *Let $p > 2$ be a prime, and let $(\lambda', f') \in GT_p$ be such that $\lambda' \in 1 + p\mathbb{Z}_p^*$ and f' satisfies (24). Then, the element* $(1, \psi) = \ln(\lambda', f')/\ln(\lambda') \in \mathfrak{gt}(\mathbb{Q}_p)$ *is of the form*

$$\psi \in \mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle + \sum_{s=0}^{\infty} p^{-(s+1)}\mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq p^s(p-1)}. \tag{25}$$

Proof Using the Taylor series expansion of $\ln(\lambda', f')$, we obtain

$$\psi = \frac{1}{\ln(\lambda')} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \psi_n,$$

where $\psi_1(x, y) = f'(x, y) - 1$, and

$$\begin{aligned}\psi_n(x, y) &= \psi_{n-1}(\lambda f' x f'^{-1}, \lambda y)\, f(x, y) - \psi_{n-1}(x, y)\\ &= \psi_{n-1}(\lambda f' x f'^{-1}, \lambda y)\, \psi_1(x, y) + \Big(\psi_{n-1}(\lambda f' x f^{-1}, \lambda y) - \psi_{n-1}(x, y)\Big).\end{aligned}$$

Note that the transformation $x \mapsto \lambda x$ corresponds to

$$\hat{x} \mapsto (1 + \hat{x})^{\lambda} - 1 \equiv (1 + p)\hat{x} \mod (p^2\hat{x}, p\hat{x}^2, \hat{x}^p),$$

and that $\psi_1 \in p\mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle + \mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq p-1}$. By induction on n, we verify that

$$\psi_n \in \sum_{k=0}^{n} p^k \mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq (n-k)(p-1)}.$$

Thus, we have

$$\psi \in \sum_{n=1}^{\infty}\sum_{k=0}^{n} n^{-1} p^{k-1}\mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq (n-k)(p-1)}.$$

Let $s+1 = 1-k+v_p(n)$ be the exponent of p in the denominator. For $s \geq 0$, we have $v_p(n) = k+s$ and $n \geq p^{k+s}$. Then, $n-k$ takes its minimal value p^s at $k=0$. This yields

$$\psi \in \mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle + \sum_{s=0}^{\infty} p^{-(s+1)} \mathbb{Z}_p\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq p^s(p-1)},$$

as required. □

Remark A.1 When viewed as a power series in $\hat{x}$ and $\hat{y}$, the element ψ in Proposition A.1 is convergent in the p-adic topology provided p-adic valuations $v_p(\hat{x})$, and $v_p(\hat{y})$ are strictly positive. As a power series in x and y, it converges provided $\hat{x}$ and $\hat{y}$ are convergent power series, i.e., if $v_p(x), v_p(y) > 1/(p-1)$.

In the case of $p=2$, we get the following result.

Proposition A.2 *Let* $(\lambda, f) \in GT_2$ *be such that* $\lambda \in 1+4+8\mathbb{Z}_2$. *Then, the element* $(1, \psi) = \ln(\lambda, f)/\ln(\lambda) \in \mathfrak{gt}(\mathbb{Q}_2)$ *is of the form*

$$\psi \in \sum_{s=0}^{\infty} 2^{-(s+2)} \mathbb{Z}_2\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq 2^s}. \tag{26}$$

Proof The degree one contribution in f vanishes. Hence, for $(\lambda, f) \in \mathrm{GT}_2$, we have $f \in 1+\mathbb{Z}_2\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq 2}$. That is, $\psi_1 = f-1 \in \mathbb{Z}_2\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq 2}$. Similarly to the previous proposition, we obtain

$$\psi_n \in \sum_{k=0}^{n} 2^k \mathbb{Z}_2\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq n-k},$$

and

$$\psi \in \sum_{n=1}^{\infty}\sum_{k=0}^{n} n^{-1} 2^{k-2} \mathbb{Z}_2\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq n-k}.$$

Here, we used the fact that $\ln(\lambda) \in 4\mathbb{Z}_2^*$. By putting $s+2 = v_2(n)+2-k$, we get $v_2(n) = k+s$, and, for $s \geq 0$, we obtain the estimate $n \geq 2^{k+s}$. Again, the minimum of $n-k$ is attained at $k=0$, which implies the desired estimate for ψ,

$$\psi \in \sum_{s=0}^{\infty} 2^{-(s+2)} \mathbb{Z}_2\langle\langle \hat{x}, \hat{y}\rangle\rangle^{\geq 2^s}.$$

Note that the study of $s=-1$ (corresponding to the first order pole) is not needed since we already have at least $1/2^2$ in all degrees. □

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